Tverberg's theorem with constraints

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Abstract

The topological Tverberg theorem claims that for any continuous map of the $(q-1)(d+1)$-simplex $\sigma^{(d+1)(q-1)}$ to $\mathbb{R}^d$ there are q disjoint faces of $\sigma^{(d+1)(q-1)}$ such that their images have a non-empty intersection. This has been proved for affine maps, and if q is a prime power, but not in general.
We extend the topological Tverberg theorem in the following way: Pairs of vertices are forced to end up in different faces. This leads to the concept of constraint graphs. In Tverberg's theorem with constraints, we come up with a list of constraints graphs for the topological Tverberg theorem.
The proof is based on connectivity results of chessboard-type complexes. Moreover, Tverberg's theorem with constraints implies new lower bounds for the number of Tverberg partitions. As a consequence, we prove Sierksma's conjecture for $d = 2$, and $q = 3$.

1 Introduction

Helge Tverberg showed in 1966 that any $(d+1)(q-1)+1$ points in $\mathbb{R}^d$ can be partitioned into q subsets such that their convex hulls have a non-empty intersection. This has been generalized to the following statement by Bárány et al. [1] for primes q, and by Özaydin [10] and Volovikov [12] for prime powers q, using the equivariant method from topological combinatorics. The general case for arbitrary q is open.

Theorem 1. *Let $q \geq 2$ be a prime power, $d \geq 1$. For every continuous map $f : \|\sigma^{(d+1)(q-1)}\| \to \mathbb{R}^d$ there are q disjoint faces $F_1, F_2, \ldots, F_q$ in the standard $(d+1)(q-1)$-simplex $\sigma^{(d+1)(q-1)}$ such that their images under f have a non-empty intersection.*

The special case for affine maps f is equivalent to the original statement of Tverberg. A partition $F_1, F_2, \ldots, F_q$ as above is a *Tverberg partition.* A point in the non-empty intersection is a *Tverberg point.* In 2005, Schöneborn and Ziegler [11, Theorem 5.8] showed that for primes p every continuous map $f : \|\sigma^{3p-3}\| \to \mathbb{R}^2$ has a Tverberg partition subject to the following type of constraints: Certain pairs of points end up in different partition sets. In other words, there is a Tverberg partition that does not use the edge connecting this pair of points.

To formalize this, let G be a subgraph of the 1-skeleton of $\sigma^{(d+1)(q-1)}$, and $f : \sigma^{(d+1)(q-1)} \to \mathbb{R}^d$ be a continuous map. Let $E(G)$ be the set of edges of G. A Tverberg partition $F_1, F_2, \ldots F_q \subset \sigma^{(d+1)(q-1)}$ of f is a *Tverberg partition of f not using any edge of G* if

$$|F_i \cap e| \leq 1 \text{ for all } i \in [q] \text{ and all edges } e \in E(G).$$

Their proof can easily be carried over to arbitrary dimension $d \geq 1$, and to prime powers q so that one obtains the following statement. A *matching* on a graph G is a set of edges of G such that no two of them share a vertex in common.

Theorem 2. *Let $q > 2$ be a prime power, and M a matching on the graph of $\sigma^{(d+1)(q-1)}$. Then every continuous map $f : \|\sigma^{(d+1)(q-1)}\| \to \mathbb{R}^d$ has a Tverberg partition $F_1, F_2, \ldots, F_q$ not using any edge from M.*

Schöneborn and Ziegler use the more general concept of winding partitions. For the sake of simplicity, we do not use this setting. However, all results in this paper also hold for winding partitions.

Theorem 2 was an important step for better understanding of Tverberg partitions: One can force pairs of points to be in different partition sets of a Tverberg partition. Choose disjoint pairs of vertices of $\sigma^{(d+1)(q-1)}$, then this choice corresponds to a matching M in the 1-skeleton of $\sigma^{(d+1)(q-1)}$. For any map f, the endpoints of any edge in M end up in different partition sets due to Theorem 2.

We extend their result to a wider class of graphs based on the following approach.

Definition. A *constraint graph C* in $\sigma^{(d+1)(q-1)}$ is a subgraph of the graph of $\sigma^{(d+1)(q-1)}$ such that every continuous map $f : \|\sigma^{(d+1)(q-1)}\| \to \mathbb{R}^d$ has a Tverberg partition of disjoint faces not using any edge from C.

Theorem 2 implies that any matching in $\sigma^{(d+1)(q-1)}$ is a constraint graph for prime powers q. Schöneborn and Ziegler [11] also come up with an example showing that the bipartite graph $K_{1,q-1}$ is not a constraint graph for arbitrary q.

The alternating drawing of K_{3q-2} is shown in Figure 1 for $q = 4$. If one deletes the first $q-1$ edges incident to the right-most vertex, then one can check that there is no Tverberg partition. In Figure 1, the deleted edges are drawn in broken lines. Numbering the vertices from right to left with the natural numbers in $[3q-2]$, the edges of the form $(1, 3q-2-2i)$, for $0 \le i \le q-2$, are deleted.

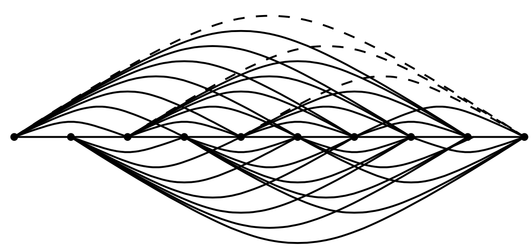

Figure 1: K_{10} minus three edges with no winding partition.

The following theorem generalizes both Theorems 1 and 2. Moreover, it implies that $K_{1,q-1}$ is a minimal example for prime powers q: All subgraphs of $K_{1,q-1}$ are constraint graphs.

Theorem 3. *Let $q > 2$ be a prime power. Then the following subgraphs of $\sigma^{(d+1)(q-1)}$ are constraint graphs:*

i) *Complete graphs K_l on l vertices for $2l < q + 2$,*

ii) *complete bipartite graphs $K_{1,l}$ for $l < q - 1$,*

iii) *paths P_l on $l + 1$ vertices for $l \le (d + 1)(q - 1)$ and $q > 3$,*

iv) *cycles C_l on l vertices for $l \le (d + 1)(q - 1) + 1$ and $q > 4$,*

v) *and arbitrary disjoint unions of graphs from* (i)–(iv).

The family of constraint graphs is closed under taking subgraphs. It is thus a monotone graph property. Theorem 3 serves us below to estimate the number of

Tverberg points in the prime power case. It is easy to see that K_2 is not a constraint graph for $q = 2$.

Figure 2 shows an example of a configuration of 13 points in the plane together with a constraint graph. Theorem 3 implies that there is a Tverberg partition into 5 blocks that does not use any of the broken edges. In Figure 2, there is for example the Tverberg partition $\{6, 10\}$, $\{9, 11\}$, $\{0, 2, 8\}$, $\{1, 5, 12\}$, $\{3, 4, 7\}$ that does not use any of the broken edges.

The constraint graph K_l guarantees that all l points end up in l pairwise disjoint partition sets. The constraint graph $K_{1,l}$ forces that the singular point in one shore of $K_{1,l}$ ends up in a different partition set than all l points of the other shore.

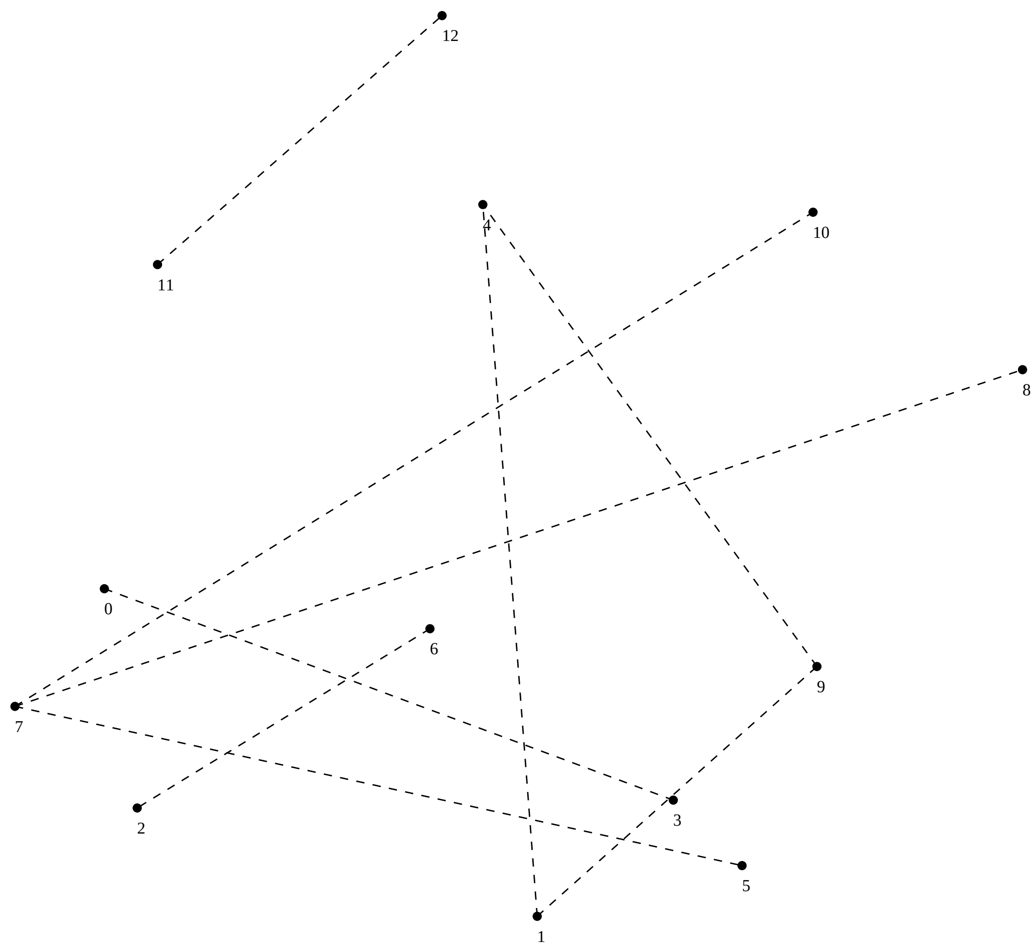

Figure 2: A planar configuration together with a constraint graph for $q = 5$.

On the number of Tverberg partitions. Tverberg's theorem establishes the existence of at least one Tverberg partition. Vućić and Živaljević [13], and Hell [7] showed that there is at least

$$\frac{1}{(q-1)!} \cdot \left(\frac{q}{r+1}\right)^{\lceil \frac{(d+1)(q-1)}{2} \rceil}$$

many Tverberg partitions if $q = p^r$ is a prime power.

Recently, Hell [5] showed a lower bound in the original affine setting of Tverberg which holds for arbitrary q.

Theorem 4. *Let X be a set of $(d+1)(q-1)+1$ points in general position in $\mathbb{R}^d$, $d \geq 1$. Then the following properties hold for the number $T(X)$ of Tverberg partitions:*

i) *$T(X)$ is even for $q > d+1$.*

ii) *$T(X) \geq (q-d)!$*

Sierksma conjectured in 1979 that the number of Tverberg partitions is at least $((q-1)!)^d$. This conjecture is unsettled, except for the trivial cases $q = 2$, or $d = 1$. Using Theorem 3 on Tverberg partitions with constraints we can improve the lower bound for the affine setting of Theorem 4 in the prime power case.

Theorem 5. *Let $d \geq 2$, and $q > 2$ be a prime power. Then there is an integer constant $c_{d,q} \geq 2$ such that every set X of $(d+1)(q-1)+1$ points in general position in $\mathbb{R}^d$ has at least*

$$\min\{(q-1)!,\, c_{d,q}(q-d)!\}$$

many Tverberg partitions. Moreover, the constant $c_{d,q}$ is monotonely increasing in q, and $c_{2,3} = 4$.

This settles Sierksma's conjecture for a wide class of planar sets for $q = 3$. Using some more effort, we entirely establish Sierksma's conjecture for $d = 2$ and $q = 3$.

Theorem 6. *Sierksma's conjecture on the number of Tverberg partitions holds for $q = 3$ and $d = 2$.*

This paper is organized as follows: Section 2 comes with a reminder of what is needed in the subsequent sections. In Section 3, we prove Theorem 3. In Section 4, we obtain the connectivity results for the chessboard-type complexes needed in Section 3. In Section 5, we prove Theorems 5 and 6.

2 Preliminaries

Let's prepare our tools from topological combinatorics, and start with some preliminaries to fix our notation, see also Matoušek's textbook [9]. Let $k \geq -1$. A topological space X is *k-connected* if for every $l = -1, 0, 1, \ldots, k$, each continuous map $f : S^l \to X$ can be extended to a continuous map $\bar{f} : B^{l+1} \to X$. Here S^{-1} is interpreted as the empty set and B^0 as a single point, so (-1)-connected means non-empty. We write $\mathrm{conn}(X)$ for the maximal k such that X is k-connected. There is an inequality for the connectivity of the join $X * Y$ for topological spaces X and Y which we use:

$$\mathrm{conn}(X * Y) \geq \mathrm{conn}(X) + \mathrm{conn}(Y) + 2; \tag{1}$$

see also [9, Section 4.4].

Deleted joins. The *n-fold n-wise deleted join* of a topological space X is

$$X^{*n}_{\Delta} := X^{*n} \setminus \{\tfrac{1}{n}x_1 \oplus \tfrac{1}{n}x_2 \oplus \cdots \oplus \tfrac{1}{n}x_n \mid \ n \text{ of the } x_i \in X \text{ are equal}\}.$$

We remove the diagonal elements from the n-fold join X^{*n}.

For a simplicial complex K we define its *n-fold pairwise deleted join* as the following set of simplices:

$$\mathsf{K}^{*n}_{\Delta(2)} := \{F_1 \uplus F_2 \uplus \cdots \uplus F_n \in \mathsf{K}^{*n} \mid F_1, F_2, \ldots, F_n \text{ pairwise disjoint}\}.$$

Both constructions show up in the proof of the topological Tverberg theorem. The p-fold pairwise deleted join of the n-simplex σ^n is isomorphic to the $n+1$-fold join of a discrete space of p points:

$$(\sigma^n)^{*p}_{\Delta(2)} \cong ([p])^{*(n+1)}. \tag{2}$$

In particular, the simplicial complex $(\sigma^n)^{*p}_{\Delta(2)}$ is n-dimensional, and $(n-1)$-connected.

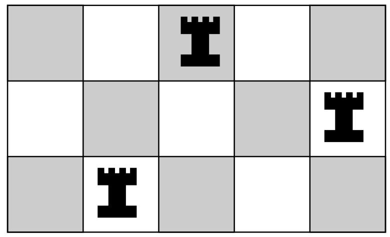

Figure 3: A maximal face of the chessboard complex $\Delta_{3,5}$.

The *chessboard complex* $\Delta_{m,n}$ is defined as the simplicial complex $([n])^{*m}_{\Delta(2)}$. Its vertex set is the set $[n] \times [m]$, and its simplices can be interpreted as placements of rooks on an $n \times m$ chessboard such that no rook threatens any other; see also Figure 3. The roles of m and n are hence symmetric. $\Delta_{m,n}$ is an $(n-1)$-dimensional simplicial complex with $\binom{m}{n} n!$ maximal faces for $m \geq n$. See also Figure 3, every maximal face corresponds to a placement of 3 rooks on a 3×5 chessboard. Having equation (2) in mind, the chessboard complex $\Delta_{n,p}$ can be seen as a subcomplex of $(\sigma^n)^{*p}_{\Delta(2)}$.

Nerve Theorem. Another very useful tool in topological combinatorics is the nerve theorem, e. g. it can be used to determine the connectivity of a given topological space, or simplicial complex. The *nerve* $N(\mathcal{F})$ of a family of sets $\mathcal{F}$ is the abstract simplicial complex with vertex set $\mathcal{F}$ whose simplices are all $\sigma \subset \mathcal{F}$ such that $\bigcap_{F \in \sigma} F \neq \emptyset$.

The nerve theorem was first obtained by Leray [8], and it has many versions; see Björner [2] for a survey on nerve theorems.

Theorem 7 (Nerve theorem)**.** *For $k \geq 0$, let $\mathcal{F}$ be a finite family of subcomplexes of simplicial complex such that $\bigcap \mathcal{G}$ is empty or $(k - |\mathcal{G}| + 1)$-connected for all non-empty subfamilies $\mathcal{G} \subset \mathcal{F}$. Then the topological space $\| \bigcup \mathcal{F} \|$ is k-connected iff the nerve complex $\|N(\mathcal{F})\|$ is k-connected.*

Using Theorem 7 and induction, Björner, Lovász, Vrećica, and Živaljević proved in [3] the following connectivity result for the chessboard complex.

Theorem 8. *The chessboard complex $\Delta_{m,n}$ is $(\nu - 2)$-connected, for*

$$\nu := \min \{m, n, \lfloor \tfrac{1}{3}(m + n + 1) \rfloor \}.$$

G-spaces and equivariant maps. Let $(G, \cdot)$ be a finite group with $|G| > 1$. A topological space X equipped with a (left) G-action via a group homomorphism $\Phi : (G, \cdot) \to (\mathrm{Homeo}(X), \circ)$ is a *G-space* (X, Φ). Here $\mathrm{Homeo}(X)$ is the group of homeomorphisms on X, the product $\circ$ of two homeomorphisms h_1 and h_2 is their composition. A continuous map f between G-spaces (X, Φ) and (Y, Ψ) that commutes with the G-actions of X and Y is called a *G-map*, or an *equivariant map*. For $x \in X$ the set $O_x = \{g\, x \,|\, g \in G\}$ is called the *orbit* of x. A G-space (X, Φ) where every O_x has at least two elements is called *fixed point free*, i. e. no point of X is fixed by all group elements.

The spaces $(\sigma^n)^{*q}_{\Delta(2)}$, $\Delta_{q,n}$, and $(\mathbb{R}^n)^{*q}_{\Delta}$ are examples of S_q-spaces, where S_q is the symmetric group on q elements. S_q acts on all three spaces via permutation of the q factors. For every subgroup H of S_q, e. g. $\mathbb{Z}_q$, or $(\mathbb{Z}_p)^r$ for prime powers $q = p^r$, an S_q-space is turned into a H-space via restriction. In fact, $(\mathbb{R}^n)^{*q}_{\Delta}$ is a fixed point free $(\mathbb{Z}_p)^r$-space for prime powers $q = p^r$, see for example Hell [7, Lemma 5].

It is one of the key steps in the equivariant method to prove that there is no G-map between two given G-spaces. It is sufficient to prove that there is no H-map between the H-spaces obtained via restriction, for a subgroup H of G. In the proof of the topological Tverberg theorem for primes q in the version of [9], this is shown for the subgroup $\mathbb{Z}_q$ via a $\mathbb{Z}_q$-index argument.

A less standard tool from equivariant topology is due to Volovikov [12]. A cohomology n-sphere over $\mathbb{Z}_p$ is a CW-complex having the same cohomology groups with $\mathbb{Z}_p$-coefficients as the n-dimensional sphere S^n. The space $(\mathbb{R}^d)^{*q}_{\Delta}$ being homotopic to the $(d+1)(q-1)-1$-sphere is an example of a cohomology $(d+1)(q-1)-1$-sphere over $\mathbb{Z}_p$, see for example Hell [7, Lemma 6].

Proposition 9 (Volovikov's Lemma)**.** *Set $G = (\mathbb{Z}_p)^r$, and let X and Y be fixed point free G-spaces such that Y is a finite-dimensional cohomology n-sphere over $\mathbb{Z}_p$ and $\tilde{H}^i(X, \mathbb{Z}_p) = 0$ for all $i \leq n$. Then there is no G-map from X to Y.*

It is the key result in [12] to obtain Theorem 1 for prime powers q.

On Tverberg and Birch partitions. For Theorems 5 and 6, we have to review some recent results for the affine setting of Tverberg's theorem. A set of points in $\mathbb{R}^d$ is *in general position* if the coordinates of all points are independent over $\mathbb{Q}$. We have chosen this quite restrictive definition of general position for the sake of its brevity, see also [11] for a less restrictive definition. We need the following reformulation of Lemma 2.7 from Schöneborn and Ziegler [11].

Lemma 10. *Let X be a set of $(d+1)(q-1)+1$ points in general position in $\mathbb{R}^d$. Then a Tverberg partition consists of:*
- *Type I: One vertex v, and $(q-1)$ many d-simplices containing v.*
- *Type II: k intersecting simplices of dimension less than d, and $(q-k)$ d-simplices containing the intersection point for some $1 < k \leq \min\{d, q\}$.*

For $d = 2$, a type II partition consists of two intersecting segments, and $q-2$ many triangles containing their intersection point. For both types, the vertex resp. the intersection point is a Tverberg point.

Let X be a set of $k(d+1)$ points in $\mathbb{R}^d$ for some $k \geq 1$. A point $p \in \mathbb{R}^d$ is a *Birch point* of X if there is a partition of X into k subsets of size $d+1$, each containing p in its convex hull. The partition of X is a *Birch partition for p*. Let $B_p(X)$ be the number of Birch partitions of X for p. If p is not in the convex hull of X, then clearly $B_p(X) = 0$.

A Tverberg partition of a set of $(d+1)(q-1)+1$ points in $\mathbb{R}^d$ is an example of a Birch partition: For a type I partition, one of the points of this set is the Tverberg point. This point plays the role of the point p, and the remaining $(q-1)(d+1)$ points are partitioned into $q-1$ subsets of size $d+1$. For a type II partition, the intersection point is the Tverberg point which plays the role of the point p, and the remaining points are again partitioned into subsets of size $d+1$. Now Theorem 4 follows from the following result from Hell [5].

Theorem 11. *Let $d \geq 1$ and $k \geq 2$ be integers, and X be a set of $k(d+1)$ points in $\mathbb{R}^d$ in general position with respect to the origin 0. Then the following properties hold for $B_0(X)$:*

i) $B_0(X)$ *is even.*

ii) $B_0(X) > 0 \implies B_0(X) \geq k!$

3 Proof of Theorem 3

Figure 4 shows all known elementary constraint graphs for $q = 5$, except for cycles on more than four vertices. In general, intersection graphs are disjoint unions of elementary constraint graphs in the 1-skeleton of σ^N. For $q = 2$, there are no constraint graphs. For $q = 3$, a single edge K_2 is the only elementary constraint graph.

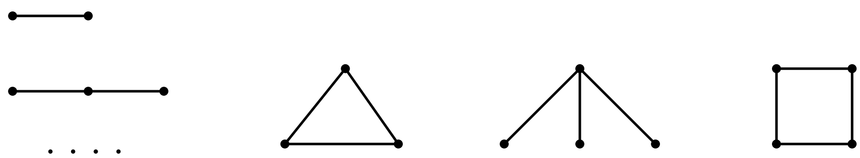

Figure 4: All known elementary constraint graphs for $q = 5$.

Proof. (of Theorem 3) Set $N := (d+1)(q-1)$, and let $q > 2$ be of the form p^r for some prime number p. As in the proof of topological Tverberg theorem in the version of [9], we consider the space $\mathsf{K} := (\sigma^N)^{*q}_{\Delta(2)}$ as configuration space. It models all possible partitions of the vertex set into q blocks: A maximal simplex of K encodes a (Tverberg) partition as shown in Figure 5, and it can be represented as a hyperedge using one point from each row of K.

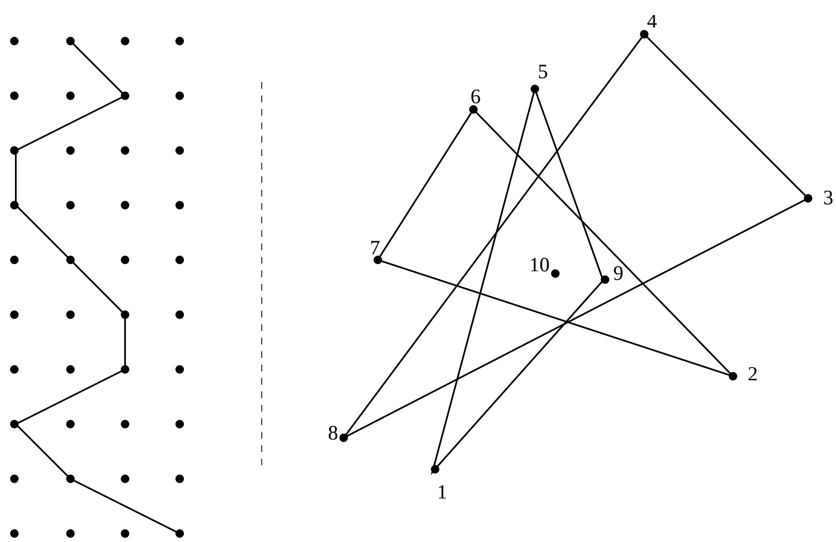

Figure 5: Maximal simplex of $(\sigma^N)^{*q}_{\Delta(2)}$ encoding a Tverberg partition.

Remember that $\|\mathsf{K}\|$ is $N-1$-connected. In the original proof of Theorem 1, the assumption that there is no Tverberg partition for f leads to the existence of a $(\mathbb{Z}_p)^r$-map $f^q : \|\mathsf{K}\| \to (\mathbb{R}^d)^{*q}_{\Delta}$. However, there is not such a map due to Volovikov's Lemma 9. Hence a Tverberg partition exists for f.

In the following, we construct for each graph a good subcomplex L of K such that: i) L is invariant under the $(\mathbb{Z}_p)^r$-action, and ii) $\mathrm{conn}(\mathsf{L}) \geq N-1$. Here *good*

means that L does not contain any of Tverberg partitions using an edge of our graph. As in the subsequent paragraph, the assumption that there is no Tverberg partition leads to a $(\mathbb{Z}_p)^r$-map $f^q : \|\mathsf{L}\| \to (\mathbb{R}^d)^{*q}_{\Delta}$. Finally Volovikov's Lemma 9 implies a contradiction, and so that there is a Tverberg partition not using any edge of our graph. Hence, our graph is a constraint graph.

Our construction of good subcomplexes is based in its simplest case – for K_2 – on the following observation:

> If two points i and j end up in the same partition set, then the maximal face representing this partition uses one of the vertical edges between the corresponding rows i and j in K.

To prove the K_2 case, we have to come up with a subcomplex L that does not contain maximal simplices using vertical edges between rows i and j. Let L be the join of the chessboard complex $\Delta_{2,q}$ on rows i and j, and the remaining rows. Figure 6 shows this construction of L for $q = 3$ and $d = 2$. The chessboard complex $\Delta_{2,q}$ does not contain any vertical edges. Moreover, L is $(\mathbb{Z}_p)^r$-invariant as only the orbit of the vertical edges is missing. For the connectivity of L see the next paragraph.

i) Construction of L for complete graphs K_l: Let $i_1, i_2, \ldots, i_l$ be the corresponding rows of K. L must not contain any maximal faces with vertical edges between any two of these rows. The chessboard complex on these rows is such a candidate. Let L be the join of the chessboard complex $\Delta_{l,q}$ on the corresponding l rows, and the remaining rows:

$$\mathsf{L} = \Delta_{l,q} * ([q])^{*(N+1-l)}.$$

The subcomplex L is closed under the $(\mathbb{Z}_p)^r$-action. Using Theorem 8 on the connectivity of the chessboard complex, and inequality (1) on the connectivity of the join, we obtain:

$$\begin{aligned}
\mathrm{conn}(\mathsf{L}) &\geq \mathrm{conn}(\Delta_{l,q}) + \mathrm{conn}(([q])^{*(N+1-l)}) + 2 \\
&\geq \mathrm{conn}(\Delta_{l,q}) + N - l + 1 \\
&\geq N - 1.
\end{aligned}$$

In the last step, we use that $\Delta_{l,q}$ is $(l-2)$-connected for $2l < q + 2$.

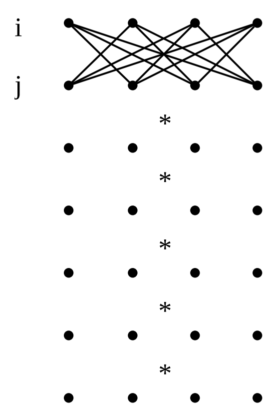

Figure 6: The construction of L for K_2.

ii) Construction of L for complete bipartite graphs $K_{1,l}$: We first construct an $(\mathbb{Z}_p)^r$-invariant subcomplex $C_{l,q}$ on the corresponding $l+1$ rows. For this, let i be the row that corresponds to the vertex of degree l, and $j_1, j_2, \ldots j_l$ be the corresponding rows to the l vertices of degree 1. Let $C_{l,q}$ be the maximal induced subcomplex of

K on the rows $i, j_1, j_2, \ldots, j_l$ that does not contain any vertical edges starting at a vertex of row i. Then $C_{l,q}$ is the union of q many complexes $L_1, L_2, \ldots, L_q$, which are all of the form of $\text{cone}([q-1]^{*l})$. Here the apex of L_m is the mth vertex of row i for every $m = 1, 2, \ldots, q$. In Figure 7, the maximal faces of the complex L_3 are shown for $q = 4$, and $l = 2$.
Let L be the join of the complex $C_{l,q}$ and the remaining rows of K:

$$\mathsf{L} = C_{l,q} * ([q])^{*(N-l)}.$$

Now L is good and $(\mathbb{Z}_p)^r$-invariant by construction. Let's assume

$$\text{conn}(C_{l,q}) \geq l - 1 \tag{3}$$

for $1 < l < q - 1$. The connectivity of L is then shown as above:

$$\begin{aligned} \text{conn}(\mathsf{L}) &\geq \text{conn}(C_{l,q}) + \text{conn}(([q])^{*(N-l)}) + 2 \\ &\geq \text{conn}(C_{l,q}) + N - l \\ &\geq N - 1. \end{aligned}$$

We prove assumption (3) in Lemma 12 below.

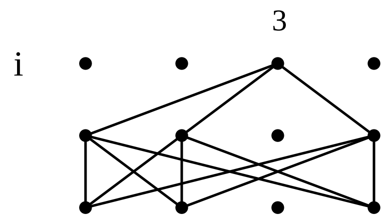

Figure 7: The complex L_3 for $q = 4$ and $l = 2$.

iii) Construction of L for paths P_l on $l+1$ vertices: We construct recursively a good subcomplex L on $l+1$ rows such that $\text{conn}(\mathsf{L}) \geq l - 1$. The case $l = 1$ is covered in the proof of i) so that we can choose L to be the complex $D_{2,q} := \Delta_{2,q}$. For $l > 1$, choose L to be the complex $D_{l,q}$ which is obtained from $D_{l-1,q}$ in the following way: Order the corresponding rows $i_1, i_2, \ldots, i_{l+1}$ in the order they occur on the path. Take $D_{l-1,q}$ on the first l rows. A maximal face F of $D_{l-1,q}$ uses a point in the last row i_l in column j, for some $j \in [q]$. We want $D_{l,q}$ to be good so that we cannot choose any vertical edges between row i_l and i_{l+1}. Let $D_{l,q}$ be defined through its maximal faces: All faces of the form $F \uplus \{k\}$ for $k \neq j$. Let $D^k_{l,q}$ be the subcomplex of all faces $D_{l,q}$ ending with k. Then $D_{l,q} = \bigcup_{k=1}^{q} D^k_{l,q}$. In Figure 8 the recursive definition of the complex $D^2_{l,5}$ is shown. The complex is

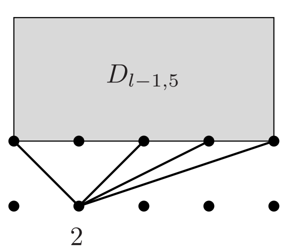

Figure 8: Recursive definition of $D^2_{l,5}$.

$(\mathbb{Z}_p)^r$-invariant, and the connectivity of $D_{l,q}$

$$\text{conn}(D_{l,q}) \geq l - 1$$

is shown in Lemma 13 below using the decomposition $\bigcup_{k=1}^{q} D_{l,q}^{k}$.

iv) Construction of L for cycles C_l on l vertices: Choose L to be the complex $E_{l,q}$ obtained from $D_{l-1,q}$ on l rows by removing all maximal simplices that use a vertical edge between first and last row. The following result on the connectivity of $E_{l,q}$ is shown in Lemma 14 below:

$$\mathrm{conn}(E_{l,q}) \geq l - 2.$$

v) Construction of L for disjoint unions of constraint graphs: For every graph component construct a complex on the corresponding rows as above. Let L be the join of these subcomplexes, and of the remaining rows. Then L is a good $(\mathbb{Z}_p)^r$-invariant subcomplex by the similar arguments as above. The connectivity of L follows analogously from inequality (1) on the connectivity of the join. ☐

Remark. Figure 11 comes with an example of a configuration of seven points in the plane showing that $P_2 = K_{1,2}$ is not a constraint graph for $q = 3$. This configuration is the outcome of a computer program, see [6, Chapter 4] for details. The same program produced many planar point configurations showing that C_4 is not a constraint graph for $q = 4$.

4 Connectivity for chessboard-type complexes

The following three lemmas provide the connectivity results needed in the proof of Theorem 3. Their proofs are similar: Inductive on l, and Theorem 7 is applied to the decompositions of the corresponding complexes that were introduced in the proof of Theorem 3.

Lemma 12. *Let $q > 2$, $d \geq 1$, and set $N = (d+1)(q-1)$. Let $C_{l,q}$ be the above defined subcomplex of $(\sigma^N)^{*q}_{\Delta(2)}$ for $1 \leq l < q - 1$. Then*

$$\mathrm{conn}(C_{l,q}) \geq l - 1.$$

Proof. In our proof, we use the decomposition of $C_{l,q}$ into subcomplexes $L_1, L_2, \ldots$ L_q from above.

The nerve $\mathcal{N}$ of the family $L_1, L_2, \ldots, L_q$ is a simplicial complex on the vertex set $[q]$. The intersection of t many $L_{m_1}, L_{m_2}, \ldots, L_{m_t}$ is $[q-t]^{*l}$ for $t > 1$ so that the nerve $\mathcal{N}$ is the boundary of the $(q-1)$-simplex. Hence $\mathcal{N}$ is $(q-3)$-connected.

Let's look at the connectivity of the non-empty intersections $\bigcap_{j=1}^{t} L_{m_j}$. For $t = 1$, every L_m is contractible as it is a cone. For $1 < t < q - 1$, the space $[q-t]^{*l}$ is $(l-2)$-connected, and for $t = q - 1$ the intersection is non-empty, hence its connectivity is -1. All non-empty intersections $\bigcap_{j=1}^{t} L_{m_j}$ are thus $(l-t)$-connected. The $(l-1)$-connectivity of $C_{l,q}$ immediately follows from the nerve theorem using $q > 2$, and $l < q - 1$. ☐

Lemma 13. *Let $q > 3$, $d \geq 1$, and set $N = (d+1)(q-1)$. Let $D_{l,q}$ be the above defined subcomplex of $(\sigma^N)^{*q}_{\Delta(2)}$ for $l \leq N$. Then*

$$\mathrm{conn}(D_{l,q}) \geq l - 1.$$

Proof. In our proof, we use the decomposition of $D_{l,q}$ into subcomplexes $D_{l,q}^1, D_{l,q}^2, \ldots, D_{l,q}^q$ from above. We prove the following connectivity result by an induction on $l \geq 1$:

$$\mathrm{conn}\Big(\bigcup_{j \in S} D_{l,q}^{j}\Big) \geq l - 1, \quad \text{for any } \emptyset \neq S \subset [q]. \tag{4}$$

Let $l = 1$, then $D_{1,q} = \bigcup_{j\in[q]} D^j_{1,q}$ is the chessboard complex $\Delta_{2,q}$ which is 0-connected for $q > 2$. The union of complexes $D^i_{1,q}$ is a union of contractible cones which is 0-connected. For $l \geq 2$, look at the intersection of $t > 1$ many complexes $D^i_{l,q}$. Let $T \subset [q]$ be the corresponding index set of size $1 < t < q-1$, and $\bar{T}$ its complement in $[q]$. Then their intersections are

$$\bigcap_{j\in T} D^j_{l,q} = \bigcup_{j\in\bar{T}} D^j_{l-1,q}\,, \tag{5}$$

$$\bigcap_{j\in[q]\setminus\{k\}} D^j_{l,q} = D^k_{l-1,q} \cup D^k_{l-2,q}\,, \quad \text{and} \tag{6}$$

$$\bigcap_{j\in[q]} D^j_{l,q} = \bigcup_{j\in[q]} D^j_{l-2,q}. \tag{7}$$

The nerve $\mathcal{N}$ of the family $D^1_{l,q}, D^2_{l,q}, \ldots, D^q_{l,q}$ is a simplicial complex on the vertex set $[q]$. The nerve is the $(q-1)$-simplex, which is contractible.

For $l = 2$, let's apply the nerve theorem. For this, we have to check that the non-empty intersection of any $t \geq 1$ complexes is $(2-t)$-connected. Every $D^j_{2,q}$ is 1-connected as it is a cone. The intersection of $t = 2$ many complexes is 0-connected for $q > 3$ by equation (5). Note that this is false for $q = 3$. The intersection of $t = 3$ many complexes is non-empty.

For $l = 3$, we have to show that the non-empty intersection of any t complexes is $(3-t)$-connected. Every $D^j_{3,q}$ is 2-connected as it is a cone. The intersection of $t < q-1$ many complexes is 1-connected by equation (5). The intersection of $t = q-1$ many complexes is a union of two cones due to equation (6). The intersection of these two cones is:

$$D^k_{2,q} \cap D^k_{1,q} = [q] \setminus \{k\},$$

which is non-empty. Using the nerve theorem, we obtain for their union:

$$\operatorname{conn}(D^k_{2,q} \cup D^k_{1,q}) \geq 0 \geq 3 - (q-1) \quad \text{for } q \geq 4.$$

The intersection of $t = q \geq 4$ many complexes is non-empty by equation (7).

Let now $l > 3$, we apply again the nerve theorem to obtain inequality (4). It remains to check that the non-empty intersection of any t complexes is $(l-t)$-connected. The complex $D^j_{l,q}$ is $(l-1)$-connected as it is a cone for every $j \in [q]$. The intersection of any $1 < t < q-1$ complexes is $(l-2)$-connected by equation (5) and by assumption. The intersection of $t = q-1$ many complexes is a union of two cones due to equation (6). The intersection of these two cones is:

$$D^k_{l-1,q} \cap D^k_{l-2,q} = \bigcup_{j\in[q]\setminus\{k\}} D^j_{l-3,q},$$

which is $(l-4)$-connected by assumption. Using the nerve theorem, we obtain for their union:

$$\operatorname{conn}(D^k_{l-1,q} \cup D^k_{l-2,q}) \geq l-3 \geq l-(q-1) \quad \text{for } q \geq 4.$$

The intersection of q many complexes is $(l-3)$-connected by equation (7) and by assumption. □

Lemma 14. *Let $q > 4$, $d \geq 1$, and set $N = (d+1)(q-1)$. Let $E_{l,q}$ be the above defined subcomplex of $(\sigma^N)^{*q}_{\Delta(2)}$ for $l \leq N+1$. Then*

$$\operatorname{conn}(E_{l,q}) \geq l-2.$$

Proof. The proof is similar to the proof of Lemma 13. The case $l = 3$ has already been settled in the proof of case i) of Theorem 3. The cases $l = 4, 5$ are analogous for $q \geq 5$, but need some tedious calculations. Observe that the inductive argument in the proof of Lemma 13 also works for $E_{l,q}$, which was obtained from $D_{l-1,q}$ by removing some maximal faces.

Let's describe the differences to the proof of Lemma 13. We consider the decomposition $E^1_{l,q}, E^2_{l,q}, \ldots, E^q_{l,q}$ of $E_{l,q}$. Here $E^i_{l,q}$ is the complex that is obtained from $D^i_{l-1,q}$ by removing all maximal faces that contain the ith vertex of the first row. In Figure 9 the complex $E^1_{l,5}$ is shown: Any face of $D^1_{l-1,q}$ containing one of the broken edges is removed.

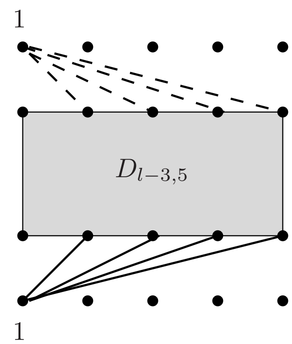

Figure 9: Subcomplex $E^1_{l,5}$ of $D^1_{l-1,5}$.

The intersection of this family is non-empty, in fact:

$$\bigcap_{i=1}^{q} E^i_{l,q} = D_{l-4,q} \quad \text{for } q \geq 5. \tag{8}$$

Thus its nerve is a simplex. Using the nerve theorem it remains to show that the intersection of $t \geq 1$ complexes is $(l-2-t+1)$-connected. For $t = 1$, the complex $E^i_{l,q}$ is a cone. For $t = q$, this follows from equation (8). For $1 < t < q$, this follows as in the proof of Lemma 13 from the equations:

$$\bigcap_{i \in [q]\setminus\{k\}} E^i_{l,q} = \tilde{D}^{k,[q]\setminus\{k\}}_{l-2,q} \cup \tilde{D}^{k,[q]\setminus\{k\}}_{l-3,q} \text{ , and} \tag{9}$$

$$\bigcap_{i \in T}^{q} E^i_{l,q} = \bigcup_{i \in \bar{T}} \tilde{D}^{i,T}_{l-2,q} \quad \text{for } T \subset [q] \text{ and } 1 < |T| < q-1, \tag{10}$$

where $\tilde{D}^{i,S}_{l,q}$ is the following subcomplex of $D^i_{l,q}$ for $S \subset [q]$: Delete all faces that contain a vertex in S of the first row. In other words $\tilde{D}^{i,\{i\}}_{l,q} = E^i_{l+1,q}$, see also Figure 10 for equation (10). There any face containing a broken edge is deleted from $D^i_{l,q}$.

Using again the nerve theorem, one then shows the necessary connectivity results for equations (9) and (10). This can be done for $q \geq 5$, inductively on $l \geq 5$:

$$\operatorname{conn}(\tilde{D}^{k,[q]\setminus\{k\}}_{l-2,q} \cup \tilde{D}^{k,[q]\setminus\{k\}}_{l-3,q}) \geq l-4,$$

and for $T \subset [q]$, $1 < |T| < q-1$:

$$\operatorname{conn}(\bigcup_{i \in \bar{T}} \tilde{D}^{i,T}_{l-2,q}) \geq l-3, \text{ and } \operatorname{conn}(\bigcup_{i \in T} \tilde{D}^{i,T}_{l-2,q})) \geq l-3.$$

$\square$

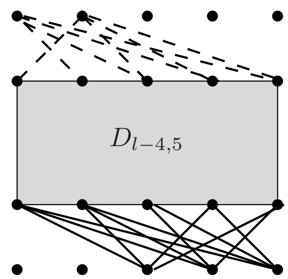

Figure 10: Equation (10): $\bigcap_{i\in\{1,2\}}^q E_{l,5}^i = \bigcup_{i\in\{3,4,5\}} \tilde{D}_{l-2,5}^{i,\{1,2\}}$

5 On the number of Tverberg partitions

In this section, we start with the proof of Theorem 5. In the proof we apply Theorem 3 on Tverberg partitions with constraints. Using a similar approach, we then settle Sierksma's conjecture for $d = 2$ and $q = 3$.

Having Theorem 11 in mind, we rise the following question:

> Is there a non-trivial lower bound for the number of Tverberg points?

In general, the answer is NO. Sierksma's well–known point configuration has exactly one Tverberg point which is of type I. This together with Theorem 11 leads to the term $(q-1)!$ in the lower bound of Theorem 5. But under the assumption that there are no Tverberg points of type I, we obtain a non-trivial lower bound for the number of Tverberg points. The constant $c_{d,q}$ is in fact a lower bound for the number of Tverberg points, assuming that there is none of type I. The factor $(q-d)!$ is due to the fact that we cannot predict what kind of type II partition shows up.

Proof. (of Theorem 5) Let X be a set of $(d+1)(q-1)+1$ points in $\mathbb{R}^d$, and p_1 is a Tverberg point which is not of type I. The Tverberg point p_1 is the intersection point of $\bigcap_{i=1}^k \operatorname{conv}(F_i^1)$, where $k \in \{2, 3, \ldots, d\}$. Choose an edge e_1 in some F_i, and apply Theorem 3 with constraint graph $G_1 = \{e_1\}$. Then there is a Tverberg partition that does not use the edge e_1 so that there has to be second Tverberg point p_2. Now add another edge e_2 from the corresponding F_i^2 to the constraint graph G_1, and apply again Theorem 3 with constraint graph $G_2 = \{e_1, e_2\}$. Hence there is another Tverberg point p_3 and so on. This procedure depends on the choices of the edges, and whether G_i is still a constraint graph.
Figure 11 shows an example for $d = 2$ and $q = 3$: A set of seven points in $\mathbb{R}^2$. There are exactly four Tverberg points – highlighted by small circles – in this example. A constraint graph – drawn in broken lines – can remove only three among them.
Constraint graphs for q are also constraint graphs for the subsequent prime power q' so that our constant $c_{d,q}$ is weakly increasing in q. The constant $c_{d,q}$ also depends on d as the simplex $\sigma^{(d+1)(q-1)}$ grows in d.

It remains to prove $c_{2,3} > 3$. For this, suppose we have three Tverberg partitions of type II for the set $\{a, b, c, d, e, f, g\}$ of seven points in $\mathbb{R}^2$.

If some edge, e. g. $\{a, b\}$, belongs to two partitions, we could find an edge in the third partition disjoint with $\{a, b\}$. The union of these two edges is a constraint graph.

If no edge belongs to two partitions, we have up to permutation the Tverberg partitions $\{a, b, c\}, \{d, e\}, \{f, g\}$ and $\{a, d, f\}, \{b, e\}, \{c, g\}$and the third partition could be either $\{a, e, g\}, \{b, d\}, \{c, f\}$ or $\{b, d, g\}, \{a, e\}, \{c, f\}$. In the former case the constraint graph $\{b, c\}, \{d, f\}, \{e, g\}$ contains an edge from every partition,

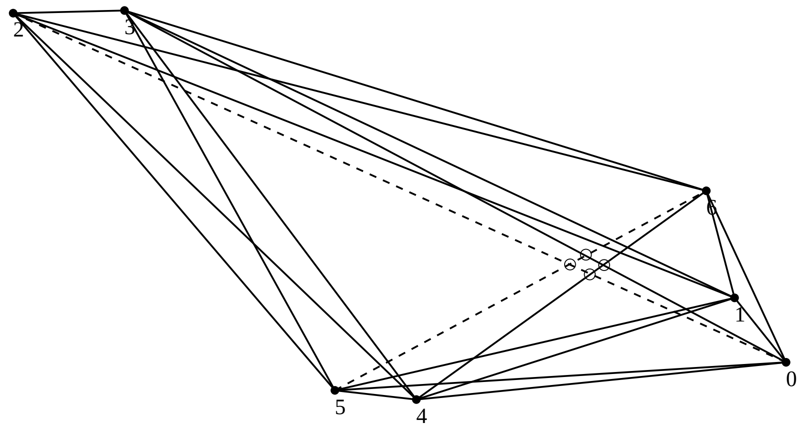

Figure 11: A set of 7 points in the plane together with a constraint graph.

and shows that there has to be a fourth Tverberg partition. In the later case, the same is true for the graph $\{b, c\}, \{a, f\}, \{d, g\}$. $\square$

Up to now, we have not been able to determine the exact value of $c_{d,q}$ for $d > 2$ or $q > 3$, as there are just too many configurations to look at. A similar – in general smaller – constant exists in the setting of the topological Tverberg theorem.

On Sierksma's conjecture. For $d = 2$ and $q = 3$, Theorem 5 settles Sierksma's conjecture for sets having no type I partition. $c_{2,3} = 4 = ((q-1)!)^d$ implies that there are at least four different Tverberg partitions. It remains to show Sierksma's conjecture for planar set of seven points having i) only type I partitions, and ii) for sets with both partition types.

Proof. (of Theorem 6) Case i). There is at least one Tverberg point coming with two partitions due to Theorem 11. It remains to show that there is one more Tverberg partition, as evenness implies the existence of the missing fourth one. Let v be the Tverberg point so that $\{v\}, \{a, b, c\}, \{d, e, f\}$ forms one of the two Tverberg partitions. Then the other Tverberg partition is of the form $\{v\}, \{a, b, d\}, \{c, e, f\}$. Choosing for example the edge $\{a, b\}$ as constraint graph completes our proof. This is not the only possible choice for G.

Case ii). There is again at least one Tverberg point v coming with two partitions of type I: $\{v\}, \{a, b, c\}, \{d, e, f\}$ and $\{v\}, \{a, b, d\}, \{c, e, f\}$. The edge $\{a, b\}$ belongs to both of these partitions. In the third partition of type II, the points a and b could belong to two sets of the partition. Choose any edge from the third set of this partition. It is disjoint with the edge $\{a, b\}$, and together with it forms the constraint graph showing that there has to be a fourth Tverberg partition. $\square$

Final remarks

Let's end with a list of problems on possible extensions of our results. The first problem aims in the direction of finding similar good subcomplexes. The second problem asks whether it is possible to show the Tverberg theorem with constraints for affine maps, independent of the fact that q is a prime power. Moreover, we conjecture that this method can be adapted to the setting of the colorful Tverberg theorem.

Problem. Determine the class $\mathcal{CG}_{q,d}$ of constraint graphs. Find graphs that are not constraint graphs. Which of the constraint graphs are maximal?
Show that cycles C_l are constraint graphs for $q = 4$, and $l \geq 5$.

Problem. Identify constraint graphs for arbitrary $q \geq 2$, especially for affine maps.

Problem. Find good subcomplexes in the configuration space $(\Delta_{2q-1,q})^{*d+1}$ of the colored Tverberg theorem to obtain a lower bound for the number of colored Tverberg partitions, and a colored Tverberg theorem with constraints.

Here a *good* subcomplex $(\Delta_{2q-1,q})^{*d+1}$ is again $(\mathbb{Z}_p)^r$-invariant, and at least $((d+1)(q-1)-1)$-connected. Constructing good subcomplexes in this setting requires more care than for the topological Tverberg theorem. One possibility to construct good subcomplexes is to identify $d+1$ many $(\mathbb{Z}_p)^r$-invariant subcomplexes L_i in the chessboard complex $\Delta_{2q-1,q}$ such that

$$\sum_{i=1}^{d+1} \operatorname{conn}(\mathsf{L}_i) \geq (d+1)(q-3)+1.$$

The join of the L_i's is then a good subcomplex in $(\Delta_{2q-1,q})^{*d+1}$. Looking at the proof for the connectivity of the chessboard complex, and studying $\Delta_{2q-1,q}$ for small q via the mathematical software system polymake [4], suggests that one obtains subcomplexes L_i by removing a non-trivial number of orbits of maximal faces.

The last problem was suggested to me by Gábor Simonyi.

Problem. Identify constraint hypergraphs.

Here a constraint hyperedge is a set of at least 3 vertices. All vertices can not end up in the same block, but any subset can. Forbidding a hyperedge of n vertices is therefore weaker than forbidding a complete graph K_n.

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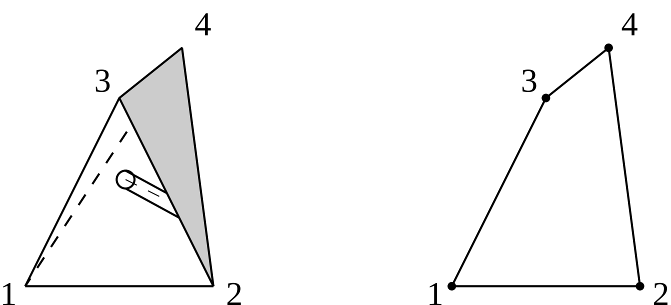

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